二. Analytic, Exponential & Harmonic Function

1. Definition of Analytic Function

Let $f : \Omega \mapsto \mathbb{C}$ where $\Omega \subseteq C$.

f is complex-differentiable at z_0 if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lambda$$

A much easier formula is $\{z_n\} \xrightarrow{n \to \infty} z_0$

$$\lim_{n \to \infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = \lambda$$

If λ is unique, then it's called derivative of f at z_0.

If f is differentiable everywhere, it's call $\underset{\text{holomorphic}}{\underline{\text{analytic}}}$ function.

Derivative rule is the same as real function's

2. Real Differentiability

If we separate f as $f(z) = u(z) + j\,v(z)$, we can analyze $f(z)$ by $u(z), v(z)$ instead. They are real functions.

$f(z)$ is continuous at z_0 iff $u(z), v(z)$ continuous at z_0

<u>But!</u> if $u(z)$ & $v(z)$ are independent, $f(z)$ is usually $\boxed{\text{non-analytic!}}$

Because $\lim\limits_{\text{from real}} f(z) = \lim\limits_{z \to z_0} u(z) \neq \lim\limits_{z \to z_0} v(z) = \lim\limits_{\text{from imagine}} f(z)$

Def:

$g: \Omega \mapsto \mathbb{R}$ Ω is an open set in $\mathbb{C}$.

g is real-differentiable at $z_0 = x_0 + jy_0 \in \Omega$ if

$\exists A, B \in \mathbb{R}$, a neighbourhood $N_\varepsilon(x_0, y_0)$, two continuous functions $\varepsilon_1(\cdot,\cdot)$ & $\varepsilon_2(\cdot,\cdot)$ s.t.

$$\varepsilon_1(x_0, y_0) = \varepsilon_2(x_0, y_0) = 0, \text{ and}$$

$$g(x,y) = g(x_0, y_0) + (x - x_0)[A + \varepsilon_1(x,y)] + (y - y_0)[B + \varepsilon_2(x,y)]$$

$\forall (x,y) \in N_\varepsilon(x_0, y_0)$

This definition looks like partial derivative. So, we further define:

Def:

If g is real-differentiable at (x_0, y_0), then the partial derivative exists at (x_0, y_0) and

$$\frac{\partial}{\partial x} g \Big|_{(x_0, y_0)} = A \quad \frac{\partial}{\partial y} g \Big|_{(x_0, y_0)} = B$$

Lemma

$\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$ exists at (x_0, y_0) and at least one of it exists at $N_\varepsilon(x_0, y_0)$, and also continuous at (x_0, y_0), $\Rightarrow$ g is real differentiable at (x_0, y_0).

Proof:

Assume $\frac{\partial}{\partial x} g$ exists at (x_0, y_0), and in $N_\varepsilon(x_0, y_0)$ it's a continuous function $A(x, y)$, then

$$g(x,y) - g(x_0,y_0) = g(x,y) - g(x_0,y) + g(x_0,y) - g(x_0,y_0)$$

Use Lagrangian Mean Value Theorem

$\exists \xi \in (x_0, x)$ s.t.

$$g(x,y) - g(x_0,y) = A(\xi)\cdot(x - x_0)$$ ✓

$\because \frac{\partial}{\partial y} g \Big|_{(x_0,y_0)} = B$

$$\therefore g(x_0,y) - g(x_0,y_0) = [B + o(y_0 - y)]\cdot(y - y_0)$$ ✓

3. Cauchy - Riemann Equations

Let $f: \Omega \mapsto \mathbb{C}$. $f(z) = u(z) + jv(z)$, $u(z), v(z)$ are real

$f(z)$ is complex-differentiable at (x_0, y_0)

iff

$u(z), v(z)$ are real-differentiable at (x_0, y_0)

and Cauchy-Riemann Equations are satisfied at (x_0, y_0)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Further, if $z_0 = x_0 + jy_0$

$$f'(z_0) = \frac{\partial u}{\partial x}\Big|_{(x_0,y_0)} + j\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)} = \frac{\partial v}{\partial y}\Big|_{(x_0,y_0)} - j\frac{\partial u}{\partial y}\Big|_{(x_0,y_0)}$$

Proof $\Rightarrow$

f is complex-differentiable at z_0, $\varepsilon(x,y)$ is defined as

$$Re[\varepsilon(x,y)] = \varepsilon_1(x,y)$$

$$Im[\varepsilon(x,y)] = \varepsilon_2(x,y)$$

for $f(z) = f(z_0) + [f'(z_0) + \varepsilon(z)](z - z_0)$

if we take real parts.

$$\frac{\partial u}{\partial x} + \varepsilon_1(z)$$

$$u(x,y) = u(x_0, y_0) + [Re[f'(z_0)] + \varepsilon_1(z)](x - x_0) - [Im[f'(z_0)] + \varepsilon_2(z)](y - y_0)$$

$$\frac{\partial u}{\partial y} - \varepsilon_2(z)$$

$$\Rightarrow \begin{cases} \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = Re\, f'(z_0) \\ \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} = -Im\, f'(z_0) \end{cases}$$

Same trick by take the imaginary part

$$\Rightarrow \begin{cases} \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} = Im\, f'(z_0) \\ \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} = Re\, f'(z_0) \end{cases}$$

Proof $\Leftarrow$

$$u(x,y) = u(x_0,y_0) + (x - x_0)\left[\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + \varepsilon_1(x,y)\right] + (y - y_0)\left[\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} + \varepsilon_2(x,y)\right]$$

$$v(x,y) = v(x_0,y_0) + (x - x_0)\left[\left.\frac{v}{\partial x}\right|_{(x_0,y_0)} + \varepsilon_3(x,y)\right] + (y - y_0)\left[\left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} + \varepsilon_4(x,y)\right]$$

Def $\varepsilon(z) = \frac{x-x_0}{z-z_0}[\varepsilon_1(x,y) + j\varepsilon_3(x,y)]$

$$+ \frac{y-y_0}{z-z_0}[\varepsilon_2(x,y) + j\varepsilon_4(x,y)]$$

we have

$$f(z) = f(z_0) + (z-z_0)\left[\frac{\partial u}{\partial x}\Big|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)} + \varepsilon(z)\right] \checkmark$$

4. The Exponential Function.

In $\mathbb{R}$. $\exp(x) = e^x \in \mathbb{R}$.

In $\mathbb{C}$. $\exp(z) = \exp(x+jy) = e^x e^{jy}$

$$= e^x(\cos y + j\sin y) \in \mathbb{C}$$

So, exponential function can be easily extended to complex domain.

Theorem.

Exponential function is analytic on $\mathbb{C}$ & $\frac{d}{dz}\exp(z) = \exp(z)$

Proof:

$$\left.\begin{aligned} Re(\exp(z)) &= e^x \cos y \\ Im(\exp(z)) &= e^x \sin y \end{aligned}\right] \xrightarrow{\text{C-R Eq} \checkmark} \begin{cases} \frac{\partial}{\partial x} e^x \cos y = e^x \cos y = \frac{\partial}{\partial y} e^x \sin y \\ \frac{\partial}{\partial y} e^x \cos y = -e^x \sin y = -\frac{\partial}{\partial x} e^x \sin y \end{cases}$$

and it's easy to check that $e^x \cos y$ & $e^x \sin y$ are real-differentiable

So, its complex-differentiable everywhere.

5. Harmonic Function

function $g : \Omega \mapsto \mathbb{C}$ is said to be Harmonic if

$g(\cdot)$ has continuous 1st & 2nd partial-derivative on Ω

and satifies Laplace's equation

$$\frac{\partial^2}{\partial x^2} g + \frac{\partial^2}{\partial y^2} g = 0 \quad \forall z \in \Omega$$

Theorem

$u : \Omega \mapsto \mathbb{R}$ is harmonic on Ω, D is an open disk $\in \Omega$.

$\exists v : D \mapsto \mathbb{R}$ s.t. $u + jv$ is analytic on D.

$v(\cdot)$ is called harmonic conjugate of $u(\cdot)$.

Proof:

Consider the differential $Pdx + Qdy$

$$\begin{cases} P = -\dfrac{\partial u}{\partial y} \\ Q = \dfrac{\partial u}{\partial x} \end{cases}$$

$\because$ u is harmonic, so P, Q are continuous

also $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ by laplace equation.

So, $\exists v$ s.t $dv = Pdx + Qdy$

$$\Rightarrow \begin{cases} \dfrac{\partial v}{\partial x} = P = -\dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial y} = Q = \dfrac{\partial u}{\partial x} \end{cases}$$

So $u + iv$ is analytic on D